LAMINAR FLOW OF THIN LIQUID FILMS

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Theoretical and experimental papers on the hydrodynamics of laminar liquid films moving along sloping surfaces are analyzed. The Nusselt theory is the asymptotic solution of the complete system of Navier-Stokes equations.

Analysis of the motion of thin liquid films is of interest both in connection with the theory of a laminar boundary layer and in connection with a possible analytic description of the kinetics of heat and mass transfer in several absorption and fractionation systems. Among these are various types of film-type mass-transfer apparatus: absorbers with a regular (tubular or plane-parallel) packing, wetted-wall fractionation columns, etc.

The steady-state motion of thin liquid films can be described by the Navier-Stokes equations and the continuity equation with boundary conditions reflecting the adhesion of the liquid to the solid surface and the balance of the normal and tangential forces at the film surface.

If there is no dynamic interaction at the surface, and there is a macroscopic balance of the liquid, the boundary-value problem can be written in terms of dimensionless variables:

$$\theta U\frac{\partial U}{\partial X}+\theta V\frac{\partial U}{\partial Y}=-\theta\frac{\partial P}{\partial X}+\frac{1}{\mathrm{Re}}\left(\theta^2\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{1}{\mathrm{Fr}},$$

$$\theta^2 U\frac{\partial V}{\partial X}+\theta^2 V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{1}{\mathrm{Re}}\left(\theta^3\frac{\partial^2 V}{\partial X^2}+\theta\frac{\partial^2 V}{\partial Y^2}\right),$$

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0,$$

$$U|_{Y=0}=V|_{Y=0}=0,$$

$$P|_{Y=H}+\frac{\theta^2\dfrac{d^2H}{dX^2}}{\mathrm{We}\left[1+\theta^2\left(\dfrac{dH}{dX}\right)^2\right]^{3/2}}+\frac{2\theta}{\mathrm{Re}}\,\frac{1+\theta^2\left(\dfrac{dH}{dX}\right)^2}{1-\theta^2\left(\dfrac{dH}{dX}\right)^2}\,\frac{\partial U}{\partial X}\bigg|_{Y=H}=0, \tag{1}$$

$$\frac{\partial U}{\partial Y}\bigg|_{Y=H}+\theta^2\frac{\partial V}{\partial X}\bigg|-\theta^2\frac{4\dfrac{dH}{dX}}{1-\theta^2\left(\dfrac{dH}{dX}\right)^2}\,\frac{\partial U}{\partial X}\bigg|_{Y=H}=0,$$

$$\frac{\partial H}{\partial X}=\frac{V}{U}\bigg|_{Y=H}.$$

In Eqs. (1), the dimensionless variables, the functions, and their derivatives do not exceed one in order of magnitude. They are obtained with the help of

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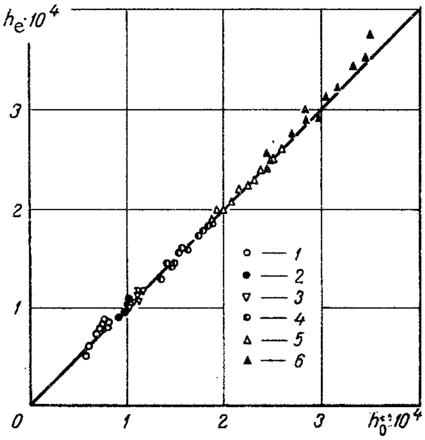

Fig. 1. Experimental and theoretical film thicknesses (in meters) for various liquids. 1) Water (the present experimental results); 2) methyl alcohol [11]; 3) water [11]; 4) isopropyl alcohol [11]; 5) aqueous solution of glycerine, 40 vol. % [11]; 6) aqueous solution of glycerine, 55 vol. % [11].

$$x = l_0 X, \ y = h_0 Y, \ u(x, y) = u_{av} U(X, Y),$$
$$v(x, y) = \theta u_{av} V(X, Y), \ p(x, y) = \rho u_{av}^2 P(X, Y), \ h(x) = h_0 H(x), \tag{2}$$

where l_0 and $h_0 = h(l_0)$ are the scale length and scale thickness of the film, and u_{av} is the average film velocity at the cross section $x = l_0$. In solving problem (1) we must add boundary conditions for two values of X. At the beginning of the film (X = 0) the liquid flows out of a slit of thickness a, and the boundary conditions are

$$X = 0, \ U = U_0(Y), \ V \equiv 0, \ P \equiv P_0, \ H = a/h_0, \tag{3}$$

where $U_0(Y)$ depends on the construction of the apparatus used to wet the wall. The entrance profile $U_0(Y)$ can be the Hagen–Poiseuille profile, the ideal-displacement profile ($U_0 \equiv h_0/a$), or any intermediate profile. The second boundary condition can be imposed on X only at $X \to \infty$. Since this condition does not follow from the physical formulation of the problem, it can be found only as the asymptotic solution of problem (1) at large X, i.e., in the zeroth approximation in the small parameter θ:

$$\frac{\partial^2 U}{\partial Y^2} = -\frac{\text{Re}}{\text{Fr}}, \quad \frac{\partial P}{\partial Y} = 0, \quad \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{4}$$
$$U|_{Y=0} = V|_{Y=0} = 0, \ P|_{Y=H} = 0, \ \left.\frac{\partial U}{\partial Y}\right|_{Y=H} = 0, \ \frac{dH}{dX} = \left.\frac{V}{U}\right|_{Y=H}.$$

The solution of problem (4) is found immediately:

$$U = \frac{\text{Re}}{\text{Fr}}\left(Y - \frac{1}{2}Y^2\right), \ V \equiv 0, \ P \equiv 0, \ H = 1. \tag{5}$$

The corresponding velocity distribution in a laminar liquid film at large x is

$$u = \frac{g}{2\nu}(2h_0 y - y^2), \ v \equiv 0. \tag{6}$$

Equation (6) was first derived by Nusselt [1, 2] as the solution of the momentum balance equation for viscous and gravitational forces in a draining film.

Analysis of problem (1) shows that the Nusselt equation, (6), is the asymptotic solution of problem (1) for large values of x; in the interval $0 \le x \le l_0$ it is necessary to solve (1) with boundary conditions (3) and (5) (X = 0 and X = 1). A length l_0 can be determined from the condition

$$\theta \sim 10^{-2}; \tag{7}$$

i.e., Eq. (6) holds within 1% for $x > l_0 \sim 10^{-2}$ m, since $h_0 \sim 10^{-4}$ m.

An experimental test of the Nusselt theory involves a test of certain basic consequences of the theory:

$$h_0 = \left(\frac{3\nu^2 \text{Re}}{g}\right)^{1/3}, \quad \frac{u_s}{u_{av}} = 1{,}5. \tag{8}$$

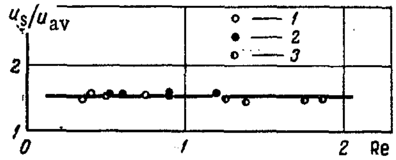

Fig. 2. Experimental ratio of the surface velocity to the average velocity of the film for various liquids. 1) Aqueous solution of diethylene glycol, 80 wt.% [12]; 2) diethylene glycol, 64 wt.% [12]; 3) glycerine, 75 wt.% [13].

Significant deviations from (8) have been observed in several experiments [3-5]; primarily, a change in the film thickness in the longitudinal direction has been observed. The existing theoretical work [6, 7] on this effect is hardly of practical value because of the ambiguity in the boundary conditions.

The velocity distribution in the initial part of the film ($0 \leq x \leq l_0$) is of only theoretical interest [8] because this part of the film is so short; the solution of this part of the problem can be found arbitrarily accurately through the use of the integral method described in [18]. Boundary value problem (1) was solved through an expansion of the functions in powers of two small parameters [10]:

$$\varepsilon_1 = \left(\frac{\theta}{\mathrm{Re}}\right)^{1/2}, \quad \varepsilon_2 = (\theta\,\mathrm{Fr})^{1/2}. \tag{9}$$

The resulting solution agrees within 0.1% with (6). The theoretical estimate in (7) has been confirmed by experiments [9], where it was shown that $l_0 < 0.02$ m.

We have now carried out an experimental test of this theory, using an experimental apparatus in which a laminar film of distilled water flows along a vertical smooth glass surface 0.5 m wide and 2 m long. In the initial part of the film the water flows out of a calibrated slit, and the wetting density is constant over the width of the film. The film thickness is measured within 10^{-6} m by a probe method. The film thickness is measured at several distances from the beginning of the film, over the interval 0.15—1.2 m, for Reynolds numbers 0.7—2. At Re > 2, waves appear at the film surface. These experiments showed that the thickness of the film is constant over its length, in complete agreement with the analogous results [11] for water, methyl and isopropyl alcohol, and aqueous solutions of glycerine. The experimental and theoretical film thicknesses are compared in Fig. 1. Statistical analysis of these results yields

$$h_e/h_0 = 1.02 \pm 0.02. \tag{10}$$

The theory can also be tested by measuring the ratio of the surface velocity to the average velocity [12, 13]; the corresponding results are shown in Fig. 2. Statistical analysis of these results yields

$$u_s/u_{av} = 1.50 \pm 0.03.$$

From Eqs. (10) and (11) we see that there is a good agreement between theory and experiment, as other experimentalists have found [14, 15].

This theoretical and experimental analysis of the flow of laminar liquid films demonstrates the accuracy and range of applicability of the Nusselt theory, as verified by the experiments of several investigators. All the experimentally observed deviations from this theory are secondary effects, which can be attributed to surface-active substances [16], gas motion [17], waves [18], etc. The accuracy of the equations for the velocity distribution [6] is completely adequate for calculating the kinetics of heat and mass transfer in laminar liquid films.

NOTATION

g) acceleration due to gravity, m/sec^2; h(x)) local film thickness, m; h_0) film thickness, m; h_e) experimental film thickness, m; l_0) film thickness, m; p(x, y)) pressure, N/m^2; u(x, y)) velocity component in the longitudinal direction, m/sec; u_{av}) average film velocity, $gh_0/3\nu$, m/sec; u_s) velocity at the film surface, $gh_0/2\nu$, m/sec; v(x, y)) transverse velocity component, m/sec; x) longitudinal coordinate, measured from the top cross section of the column, m; y) transverse coordinate, measured normally from the solid surface into the interior of the liquid film, m; Γ) wetting density, m^2/sec; θ) dimensionless parameter, equal to h_0/l_0; ν) kinematic viscosity, m^2/sec; ρ) density of liquid, kg/m^3; σ) surface tension at the liquid—gas interface, N/m; Fr) Froude number, u_{av}/gh_0; Re) Reynolds number, $\Gamma/\nu = u_{av}h_0/\nu$; We) Weber number, $\rho u_{av}h_0/\sigma$.

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